Hypersonic flows past a yawed circular cone and other pointed bodies

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A detailed treatment of inviscid hypersonic flow past a circular cone is given, for small and moderate yaw angles, within the framework of shock-layer theory. The basic problem of non-uniform validity associated with the singularity of the entropy field is examined and a valid first-order solution is obtained which provides an explicit description of a thin vortical layer at the inner edge of the shock layer. Analytic formulas for pressure and circumferential velocity are given consistent to the second-order approximation including the non-linear yaw effect.

The study of the entropy field (which is not restricted to the hypersonic case) also provides corrections to previous work on the yawed cone and confirms the validity of the linear yaw effect on pressure field in the Stone theory.

A related investigation of three-dimensional flow fields is presented with special reference to the flow structure near the surface of a pointed, but otherwise arbitrary body. The inviscid streamline pattern on the surface is given by the geodesics originating from the pointed nose as a leading approximation of shock-layer theory. Associated with this streamline pattern is a vortical sublayer which exists generally at small as well as at large angle of attack. At the base of the sublayer, enthalpy and flow speed remain essentially uniform.

1. Introduction

There exists no simple treatment for the problem of compressible flow around a body, except under the simplification for a slender body or low Mach number. For hypersonic flow involving a strong shock wave, however, an attractive approach based on the shock-layer concept is available. In the earlier work on inviscid hypersonic flow, Busemann (1933) considers the limiting situation of $M_\infty \to \infty$ and $\gamma \to 1$. In this limit, the compression ratio across the bow shock is infinitely high, the 'shock layer'—the region between the shock and the body surface—is vanishingly thin, and the result is particularly simple. Later workers, notably Ivy, Klunker & Bowen (1948), Chester (1956), Freeman (1956), and Cole (1957), have expanded the idea into a consistent treatment of plane and axisymmetric hypersonic flows. This paper is concerned with three-dimensional hypersonic flows, specifically the analysis of a yawed circular cone, and a related exploration of a few unique features of the flow field around other pointed bodies. This study, with the exception of certain portions of § 3 (which is not restricted to hypersonic flow), is carried out in the framework of the shock-layer theory.

There has been a basic difficulty in the supersonic yawed-cone problem itself. In the theory of Stone (1948, 1952), the problem is treated by a procedure of small perturbation in the yaw angle. This treatment has provided a basis for the rather extensive computations undertaken by Kopal (1947*b*, 1949). The analysis breaks down, however, in the vicinity of the cone surface. As pointed out by Ferri (1950), it fails to account for the presence of a 'vortical layer' (a thin layer of intense vorticity next to the body) and results in an incorrect value for the entropy on the surface. In the subsequent treatment of this problem, a perturbation in yaw angle will also be employed in order to gain an explicit solution to the problem. Nevertheless, the basic problem related to the singularity of the entropy field is examined and a modified scheme of approximation is found. Such inquiry is essential not only from the viewpoint of application to the boundary-layer analysis, but also for ensuring the internal consistency of both Stone's and present theories.

Using the shock-layer approach, three-dimensional problems involving a cross-flow velocity have also been treated recently by Cole (1958), Gonor (1958), Hayes & Probstein (1959), Guiraud (1959*a*), Maikapar (1959), and Laval (1959). While definite results have been obtained mostly for simple bodies, such as yawed cones and bodies of revolution at an angle of attack, the general method described by Hayes & Probstein, Maikapar and Guiraud is applicable to bodies of arbitrary shape, at least in principle. These studies pertain to the limiting situation of $M_\infty \to \infty$ and $\epsilon \equiv (\gamma - 1)/(\gamma + 1) \to 0$, and correspond to the two-dimensional theory of Busemann. In a more recent paper, Guiraud (1959*b*) has extended the treatment of the cone problem to include the first-order effect of finite M_∞ and ϵ. Within the accuracy and the limitation of the shock-layer theory, these analyses have provided useful information on the distribution of the surface pressure. In these works, the vortical layer and its effects have not been properly taken into account, although its existence may be inferred from certain peculiar features of the solutions (Gonor 1958; Guiraud 1959*b*).

While the pressure distribution on the circular cone obtained from the present treatment agrees generally with those given by other workers, the present solution provides an explicit, analytic description of the entire flow field which reveals the structure of a thin vortical layer at the inner edge of the shock layer. The approximation given for the entropy field is valid uniformly to the first order of ϵ and of the yaw angle, whereas the pressure and the circumferential velocity fields are determined consistent to the second order. The shock wave is required to be strong; the Mach number, however, is not required to be infinite. In the study pertaining to a pointed body of arbitrary shape, the streamlines near the surface are found to follow closely the surface geodesics which originate from the pointed nose. Related to this is the presence of a thin vortical layer which generally exists at small as well as at large angles of attack. To the writer's knowledge, this pattern of surface streamlines, and the related sublayer behaviour of the flow field, have not been brought out clearly in any of the previous studies, although the fact that the streamlines in the shock layer tend to follow geodesics is quite well known (Hayes & Probstein 1959; Maikapar 1959; Guiraud 1959*a*).

Recently, the writer's attention was directed to a study by Willet (1960) on supersonic flow around a yawed circular cone, in which it is shown that each inviscid streamline on the cone surface passes through the cone apex. This finding is certainly consistent with the present conclusion for the streamline pattern on a pointed body. Very recently, the vortical layer on a circular cone in supersonic flow has been studied by Woods (1960). In this work, an expression of the entropy field exactly equivalent to that based on the present scheme (Cheng 1960) is given. However, the crucial question of uniform validity of the approximation, which is handled with considerable care here, is not thoroughly examined in Woods's paper.

The numerical approach also will be mentioned. While numerical analysis of the yawed-cone problem has not met with success in the past, the inverse problem of finding the body shape corresponding to a given conical shock can be analysed numerically and has recently been studied (Radhakrishnan 1958; Briggs 1959). By cut and try, an appropriate shock shape and the corresponding solution to the flow field around the given conical surface may presumably be approached. Since a very thin vortical layer will generally exist in hypersonic flow even at large yaw angles, such numerical procedures are not satisfactory for the study of the flow field near the body surface.

Basic to the following investigation are the assumptions of a strong shock wave with a high compression ratio. For simplicity, the model of a perfect gas with constant specific heats is adopted. In all studies pertaining to pointed bodies, the shock wave is assumed to attach at the apex. The region on a convex body corresponding to the 'zero-pressure point' (Lighthill 1957; Freeman 1960), where the shock-layer approximation breaks down, is excluded from the present discussion.

Most studies presented in this paper are based on a previous report of the writer prepared under the sponsorship of the United States Air Force through the Aeronautical Research Laboratory of the Wright Air Development Division, Contract No. AF 33(616)-6025 (Cheng 1959). The analysis of the yawed-cone problem given in § 2 is less restrictive however than its previous version, in regard to the requirement on the shock strength. The writer would like to take this opportunity to thank Mrs A. L. Chang of the Aerodynamic Research Department, Cornell Aeronautical Laboratory, for her contribution to most of the detailed analyses, and Prof. H. Pollard of the Mathematics Department of Cornell University for valuable discussions. Acknowledgement is due also to Messrs J. P. Guiraud and L. Laval of O.N.E.R.A., France, who have kindly pointed out to the writer certain algebraic errors in his previous report.

2. Small-perturbation treatment of inviscid hypersonic flows over yawed circular cones

2.1. *Basic assumptions and formulation of the problem*

It is convenient to formulate the problem of a circular cone in spherical coordinates (r, ϑ, ω), where r is the distance from the origin, ϑ the angle between an axis Ox and the position vector, and ω the angle between the meridian plane (the plane containing the axis Ox and the position vector) and an x–y plane (see figure 1). The apex of the cone will be chosen as the origin, and the axis of the cone as the axis Ox. The x–y plane is normal to the cone's plane of yaw. The orthogonal velocity components along the position vector, and along the directions of increasing ϑ and ω, will be designated respectively as u_*, v_* and w_*.

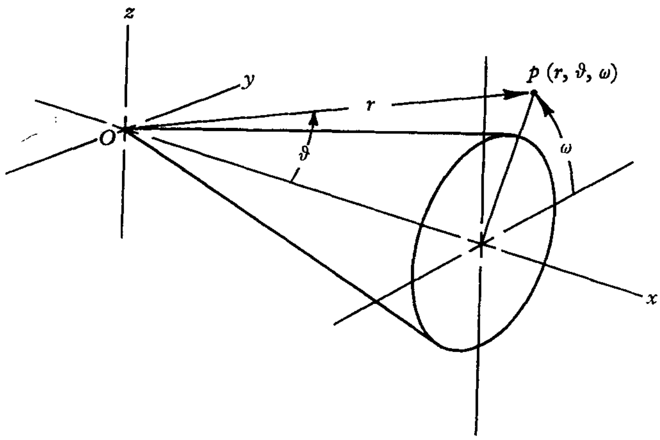

FIGURE 1. The co-ordinate system for analysing the circular-cone problem.

With the present approach to the problem in mind, a set of non-dimensional variables will be introduced to replace the physical flow quantities. Let

$$\left.\begin{aligned} u \equiv \frac{u_*}{U_\infty}, \quad v \equiv \frac{v_*}{\epsilon U_\infty \sin\tau}, \quad w \equiv \frac{w_*}{U_\infty \sin\alpha}, \\ p \equiv \frac{p_*}{\rho_* U_\infty^2 \sin^2\tau}, \quad \rho \equiv \frac{\epsilon\rho_*}{\rho_\infty}, \end{aligned}\right\} \tag{2.1}$$

and

$$\theta \equiv \frac{\sin\vartheta - \sin\tau}{\epsilon\sin\tau}, \tag{2.2}$$

where τ is the half-cone angle, p_* the pressure, and ρ_* the density. The parameter ϵ stands for $(\gamma-1)/(\gamma+1)$. When the shock-layer concept applies, the magnitudes of u, v, w, p and ρ are expected to be bounded even in the limit of vanishing ϵ. Meanwhile, the thickness of the shock layer, in terms of the new spatial variable θ, will be of order unity. The cone surface will be located at $\theta = 0$. Anticipating conical symmetry of the flow fields, the system of differential equations stating the conservation laws of momentum, mass and energy may now be written as

$$\left.\begin{aligned} \left[Jv\frac{\partial}{\partial\theta} + \sigma\frac{w}{1+\epsilon\theta}\frac{\partial}{\partial\omega}\right]u = \sin^2\tau[\epsilon^2v^2 + \sigma^2w^2], \\ \frac{\partial p}{\partial\theta} = \sigma^2\rho\frac{w^2}{1+\epsilon\theta} - \epsilon\rho\left[v\frac{\partial}{\partial\theta} + \sigma\frac{w}{J(1+\epsilon\theta)}\frac{\partial}{\partial\omega} + \frac{u}{J}\right]v, \\ \sigma\rho\left[Jv\frac{\partial}{\partial\theta} + \sigma\frac{w}{1+\epsilon\theta}\frac{\partial}{\partial\omega} + u\right]w = -\frac{\epsilon}{1+\epsilon\theta}\left[\frac{\partial p}{\partial\omega} + \sigma J\rho vw\right], \\ 2(1+\epsilon\theta)\rho u + J\frac{\partial}{\partial\theta}[(1+\epsilon\theta)\rho v] + \sigma\frac{\partial}{\partial\omega}(\rho w) = 0, \\ \left[Jv\frac{\partial}{\partial\theta} + \sigma\frac{w}{1+\epsilon\theta}\frac{\partial}{\partial\omega}\right]\left(\frac{p}{\rho^\gamma}\right) = 0, \end{aligned}\right\} \tag{2.3}$$

where σ is the yaw parameter $\sin\alpha/\sin\tau$ and J stands for $\cos\vartheta$; i.e.

$$J \equiv \{1-\sin^2\tau(1+\epsilon\theta)^2\}^{\frac{1}{2}}.$$

The boundary condition on the cone surface is simply that v vanishes at $\theta = 0$. The corresponding boundary conditions behind the shock surface are provided by the Rankine–Hugoniot relation with the upstream condition determined by the undisturbed free stream. They are

$$\left.\begin{aligned}
&J[\rho v - J\sigma\sin\omega + (1+\epsilon\theta)\cos\alpha] = \sigma\frac{\theta_\omega}{1+\epsilon\theta}[\rho w - \epsilon\cos\omega],\\
&\left[J^2+\left(\frac{\epsilon\theta_\omega}{1+\epsilon\theta}\right)^2\right](p-\epsilon\kappa) = \left[J(1+\epsilon\theta)\cos\alpha - J^2\sigma\sin\omega + \sigma\epsilon\frac{\theta_\omega}{(1+\epsilon\theta)}\cos\omega\right]^2\\
&\qquad\qquad -\epsilon\rho\left[Jv-\sigma\frac{\theta_\omega}{(1+\epsilon\theta)}w\right]^2,\\
&\left[J^2+\left(\frac{\epsilon\theta_\omega}{1+\epsilon\theta}\right)^2\right]\left(\frac{p}{\rho}-\kappa\right)(1+\epsilon)+\epsilon^2\left[Jv-\sigma\frac{\theta_\omega}{(1+\epsilon\theta)}w\right]^2\\
&\qquad\qquad = \left[J(1+\epsilon\theta)\cos\alpha - J^2\sigma\sin\omega + \sigma\epsilon\frac{\theta_\omega}{(1+\epsilon\theta)}\cos\omega\right]^2,\\
&J\sigma(w-\cos\omega)+\epsilon\frac{\theta_\omega}{(1+\epsilon\theta)}[\epsilon v - J\sigma\sin\omega + (1+\epsilon\theta)\cos\alpha] = 0,\\
&u - J\cos\alpha = \sin^2\tau(1+\epsilon\theta)\,\sigma\sin\omega,
\end{aligned}\right\}\tag{2.4}$$

where a parameter κ has been introduced to represent the constant

$$(\gamma+1)/[\gamma(\gamma-1)\,M_\infty^2\sin^2\tau].$$

The variable θ in equations (2.4) now stands for the unknown position of the shock as a function of ω, $\theta = \theta_{\text{sh}}(\omega)$, and θ_ω stands for the derivative $d\theta_{\text{sh}}/d\omega$. The first and the third of (2.4) express the conservation of mass and energy, and the second equation the conservation of momentum in the direction normal to the shock surface; the last two equations state the continuity of tangential velocity components across the shock. An integral of the differential equations, (2.3), consistent with the boundary conditions, is of course the Bernoulli relation; that is, the total enthalpy remains constant throughout the entire field. This relation may be used in place of the first of (2.3). The above system of equations will be sufficient for the determination of the fields of velocity, pressure, and density as well as the conical shock envelope.

2.2. *Analytic treatment by development in ϵ and σ*

The above formulation shows that the supersonic yawed-cone problem is reducible to one governed by four parameters, namely, τ, κ, ϵ and σ. In Stone's theory (1948, 1952), a development in ascending power of the yaw parameter σ is assumed as a basis for numerical solutions (Kopal 1947*b*). In order to provide an explicit, analytic description of the flow field, it will be tentatively assumed in the present treatment that the flow-field variables are developable in powers of both ϵ and σ. Typically

$$p = p_{00}+\epsilon p_{10}+\sigma p_{01}+\epsilon^2 p_{20}+\epsilon\sigma p_{11}+\sigma^2 p_{02}+\dots. \tag{2.5}$$

Consistent with this is a similar development for the shock position

$$\theta_{\text{sh}}(\omega) = \theta_{00} + \epsilon\theta_{10} + \sigma\theta_{01} + \ldots.$$

Since $[\theta_{\text{sh}} - \theta_{00}]$ is a controllable, small quantity, it is permissible to replace the set of outer boundary conditions, (2.4), by an equivalent one at $\theta = \theta_{00}$, consistent with the assumed expansions.

Implicit in this development are, of course, the requirements

$$\epsilon \equiv \frac{\gamma-1}{\gamma+1} \ll 1, \quad \sigma \equiv \frac{\sin\alpha}{\sin\tau} \ll 1. \tag{2.6}$$

Also necessary are some restrictions on the shock strength and the half-cone angle; namely, both $\kappa \equiv 1/\gamma\epsilon M_\infty^2 \sin^2\tau$ and $\tan\tau$ must remain bounded. The strong-shock assumption underlying the shock-layer concept is implied by the restriction on κ. The necessity for these restrictions will subsequently be apparent.†

Substitution of the expansions in ϵ and σ into the governing equations and collection of terms of equal powers, yield readily the leading approximation and the linear corrections

$$\left.\begin{aligned}
u &= \cos\tau - \epsilon\left(\frac{1+\kappa}{2}\right)\sin\tau\,\tan\tau + \sigma\sin^2\tau\,\sin\omega,\\
v &= -2\theta + \epsilon\left[\theta(1+\kappa)\tan^2\tau + \theta^2(1-\tan^2\tau) - \frac{4\theta^3}{3(1+\kappa)}\right]\\
&\qquad + \sigma\frac{\sin\omega}{\cos\tau}\left[\frac{1+\kappa}{3}\left(\frac{2\theta}{1+\kappa}\right)^{\frac{3}{2}} - 2\theta\sin^2\tau\right],\\
w &= \left(\frac{2\theta}{1+\kappa}\right)^{\frac{1}{2}}\cos\omega,\\
p &= 1 + \epsilon\left(\frac{1+5\kappa}{4} - \frac{\theta^2}{1+\kappa}\right) - \sigma 2\cos\tau\,\sin\omega,\\
\rho &= \frac{1}{1+\kappa} + \epsilon\left[\frac{1}{4} + \left(\frac{\kappa}{1+\kappa}\right)^2 - \left(\frac{\theta}{1+\kappa}\right)^2 + \frac{2}{1+\kappa}\ln(1+\kappa)\right] - \sigma\frac{2\kappa}{(1+\kappa)^2}\cos\tau\,\sin\omega,\\
\theta_{\text{sh}} &= \frac{1+\kappa}{2} + \epsilon\left[\frac{(1+\kappa)^2}{24}(7+3\tan^2\tau) - \frac{\kappa^2}{2} - (1+\kappa)\ln(1+\kappa)\right] - \sigma\frac{\sin\omega}{\cos\tau}\left[\kappa\cos^2\tau - \frac{1+\kappa}{3}\right].
\end{aligned}\right\} \tag{2.7}$$

In the same manner, but with a slight increase in the degree of complication, high-order corrections can also be determined. In particular, the circumferential velocity w can be carried to a higher order as

$$\begin{aligned}
w = \cos\omega\left(\frac{2\theta}{1+\kappa}\right)^{\frac{1}{2}} &+ \epsilon\cos\omega\Bigg\{2(1+\kappa) + \left(\frac{2\theta}{1+\kappa}\right)^{\frac{1}{2}}\\
&\times\left[\frac{(1+\kappa)}{3\cos^2\tau} - \kappa - \frac{\kappa^2}{2(1+\kappa)} - \frac{1+\kappa}{4}\tan\tau - \tfrac{15}{8}(1+\kappa) + \ln(1+\kappa)\right]\\
&+ \left(\frac{2\theta}{1+\kappa}\right)^{\frac{3}{2}}\left[\frac{(1+\kappa)}{8}\tan^2\tau - \tfrac{3}{8}(1+\kappa)\right] - \left(\frac{2\theta}{1+\kappa}\right)^{\frac{5}{2}}\frac{(1+\kappa)}{24}\Bigg\}\\
&+ \sigma\frac{\sin 2\omega}{\cos\tau}\left[\frac{1}{2}\left(1 - \frac{\kappa}{1+\kappa}\cos^2\tau\right)\left(\frac{2\theta}{1+\kappa}\right)^{\frac{1}{2}} - \frac{1}{3}\left(\frac{2\theta}{1+\kappa}\right)\right],
\end{aligned} \tag{2.8}$$

† Note that the assumptions that $\tan\tau$ is bounded and that ϵ is small ensure the existence of conical symmetry for the problem.

and the pressure field, to the second-order approximation, as

$$
\begin{aligned}
p = {} & 1+\epsilon\left(\frac{1+5\kappa}{4}-\frac{\theta^2}{1+\kappa}\right)-2\sigma\cos\tau\,\sin\omega+\epsilon^2\left[\tfrac{3}{32}(1+\kappa)^2-\frac{\kappa^2}{2}\right.\\
& -\frac{(2\theta)^2}{16}-\frac{11(2\theta)^4}{96(1+\kappa)^2}-\left(\frac{\theta\kappa}{1+\kappa}\right)^2+\frac{\tan^2\tau}{4}(1+\kappa)^2+\frac{(2\theta)^2\tan^2\tau}{4}\\
& \left.+\frac{5(2\theta)^3}{24(1+\kappa)}-\frac{\tan^2\tau(2\theta)^3}{8(1+\kappa)}-\left(\frac{2\theta^2}{1+\kappa}+\frac{1+\kappa}{2}\right)\ln(1+\kappa)\right]\\
& +\sigma^2\left[\cos 2\tau-\cos^2\omega\left(\cos^2\tau+\frac{1}{4}-\frac{\theta^2}{(1+\kappa)^2}\right)\right]+\sigma\epsilon\frac{\sin\omega}{\cos\tau}\left\{\tfrac{4}{15}(1+\kappa)\left[\left(\frac{2\theta}{1+\kappa}\right)^{\frac{5}{2}}-1\right]\right.\\
& \left.+\frac{\sin^2\tau}{2}\left[1-\frac{(2\theta)^2}{1+\kappa}\right]+\frac{\kappa}{2}\left[\left(\frac{2\theta}{1+\kappa}\right)^2\cos^2\tau-\cos 2\tau\right]\right\}.
\end{aligned}
\tag{2.9}
$$

Application of the pressure formula at the surface yields the coefficients of the normal force N and of the axial force X:

$$
\left.
\begin{aligned}
C_N &\equiv \frac{2N}{\rho_\infty U_\infty^2 \pi x^2 \sin^2\tau},\\
&= \sin\alpha\left\{2\cos^2\tau+\epsilon\left[\tfrac{4}{15}(1+\kappa)-\frac{\sin^2\tau}{2}+\frac{\kappa}{2}\cos 2\tau\right]\right\},\\
C_X &\equiv \frac{2X}{\rho_\infty U_\infty^2 \pi x^2 \sin^2\tau}\\
&= 2\sin^2\tau\left\{1+\epsilon\left(\frac{1+5\kappa}{4}\right)+\sigma^2\left(\frac{3\cos^2\tau}{2}-\frac{9}{8}\right)\right.\\
&\qquad \left.+\epsilon^2\left[\tfrac{3}{32}(1+\kappa)^2+\frac{\tan^2\tau}{4}(1+\kappa)^2-\frac{\kappa^2}{2}-\frac{1+\kappa}{2}\ln(1+\kappa)\right]\right\}.
\end{aligned}
\right\}
\tag{2.10}
$$

The remainder on the right-hand side of equations (2.8) to (2.10) consists of terms associated with the third order and higher.

The necessity of keeping the half-cone angle well below 90° is evident from the development of u given in (2.7). For the ratio of the correction term to the leading approximation is found to be $[\sigma\sin\tau\,\tan\tau\,\sin\omega-\frac{1}{2}(1+\kappa)\,\epsilon\tan^2\tau]$, which cannot remain small, as is required by the small-perturbation procedure, when $\tau\to\frac{1}{2}\pi$. Similarly, the requirement that κ be finite is also apparent from (2.7). The correction term and the leading approximation are seen to be in the proportion $O(\kappa\epsilon):1$, when κ is large.

In figure 2, the normal-force derivative $dC_N/d\alpha$ of a cone at $\alpha=0$ is evaluated from (2.10) for the particular circumstance $\epsilon\kappa\to 0$ and compared with the corresponding numerical data of Kopal (1947*b*). The numerical data presented pertains to the high Mach number range for $\gamma = 1{\cdot}405$, i.e. for $\epsilon = 0{\cdot}168$. The agreement is seen to be excellent. One may note particularly that the correction with respect to ϵ to the Newtonian result is exceedingly small. In figure 3, a comparison is made of the circumferential velocity on the cone surface approximated by (2.8) with the corresponding data computed by Kopal for $\gamma = 1{\cdot}405$. To simplify matters, only the case of slender cones (i.e. $\tau \ll 1$) is examined, for which a correlation in the parameter κ is possible. The latter parameter is essentially a variant of the hypersonic parameter for slender bodies, $M_\infty \tau$.

While the present shock-layer treatment, according to the comparisons shown, appears to yield a satisfactory approximation, the agreement of the present results and the Stone theory is by no means an indication of the validity of either

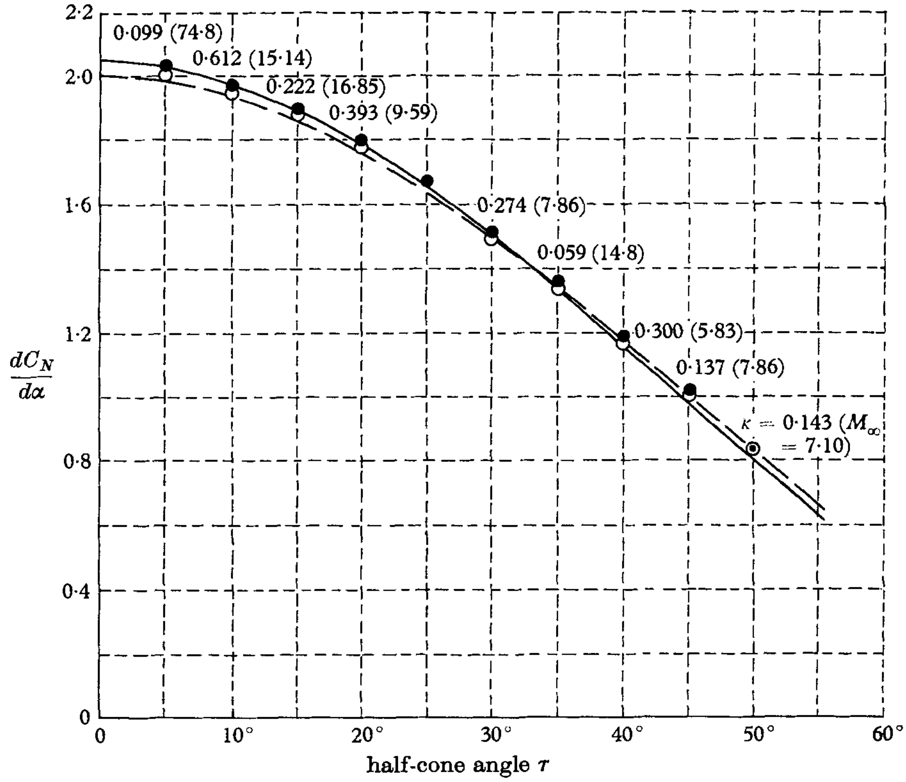

FIGURE 2. A comparison in the range of small κ of the linear lateral force based on the shock-layer theory and the corresponding data from the Kopal table. Numerical data from Kopal's table ($\gamma = 1{\cdot}405$): ●, $\kappa = 0$ ($M_\infty = \infty$); ○, κ or M_∞ as indicated. Calculation based on:

$$\text{——},\quad \left(\frac{dC_N}{d\alpha}\right)_{\alpha=0} = 2\cos^2\tau + \frac{\gamma-1}{\gamma+1}\left(\frac{4}{15} - \frac{\sin\tau}{2}\right);$$

$$\text{- - - -},\quad \left(\frac{dC_N}{d\alpha}\right)_{\alpha=0} = 2\cos^2\tau.$$

analysis. Combining formulas for the pressure and the density, the field of an entropy function $S \equiv p/\rho^\gamma$ is

$$S = 1 + K + \epsilon\kappa - 2\sigma\cos\tau\sin\omega, \tag{2.11}$$

which gives a sinusoidal variation around the surface. Since, according to (2.11), the specific entropy is independent of θ, the above result also suggests that the pattern of streamlines, when projected in the θ–ω plane, would not change with angle of attack. On the other hand, application of the last of equations (2.3) on the surface gives

$$\frac{\partial}{\partial\omega}\left(\frac{p}{\rho^\gamma}\right) = 0,$$

which indicates that the entropy must remain circumferentially uniform on the surface. Evidently, the results given in (2.11) for the entropy function, as well as other related quantities, are incorrect on the surface at least. This observation has been pointed out by Ferri (1950). The breakdown is clearly related to the invalidity of the formal development in the form of (2.5) and of the small-perturbation procedure (associated with the parameter σ) that follows. In fact, the symptom of invalidity can actually be found in the third-order approximation. Namely, as $\theta \to 0$, both ρ and u would become singular like $\epsilon\sigma^2 \ln\theta$. This should correspond to a singularity of the type $\sigma^2 \ln\theta$ in the Stone theory.†

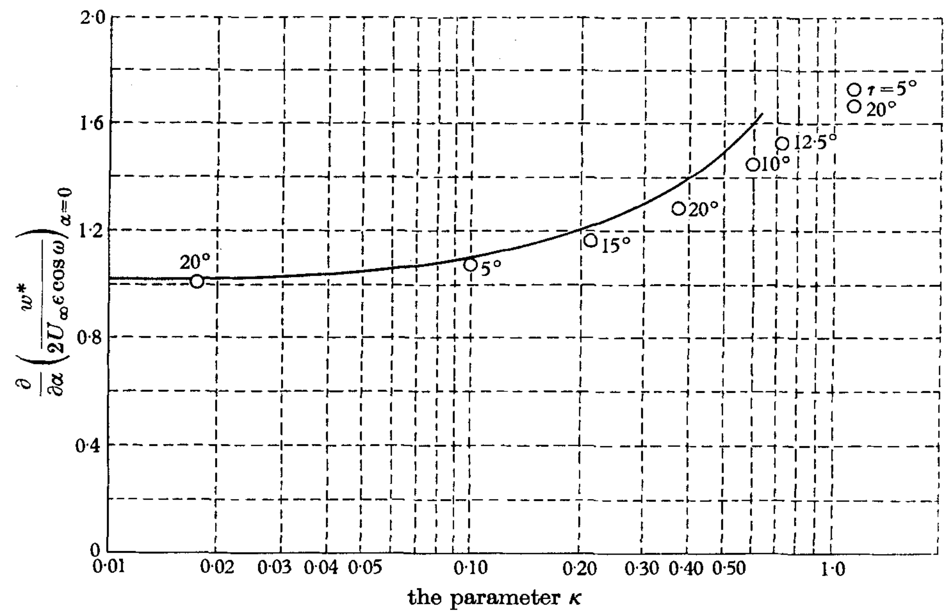

FIGURE 3. A comparison of the linear yaw effect on the circumferential velocity at the surface based on (2.8), and the corresponding data obtained from Kopal's table, for slender cones. ○, Numerical data from Kopal's table with the half-cone angles as indicated ($\gamma = 1{\cdot}405$).

$$\kappa = \frac{\gamma+1}{\gamma-1}\frac{1}{\gamma M_\infty^2 \sin^2\tau}.$$

Slender cone result $$\frac{\partial}{\partial\alpha}\left(\frac{w_*}{2U_\infty \epsilon \cos\omega}\right)_{\alpha=0} = 1+\kappa.$$

3. The sublayer behaviour of the entropy field and the modification of the small-perturbation procedure

3.1. *The singular differential equation and the scheme of approximation*

In view of its basic importance to the study of boundary layer and to the internal consistency of the yawed-cone theory itself, a critical study of the problem related to the non-uniform validity of the approximation will be made. The difficulty in the analysis will now be removed by a modified scheme which ensures

† The density field computed by Kopal (1949) shows no sign of divergence on the cone surface. This is due probably to the rather slow rate at which $\ln\theta$ approaches infinity.

uniform validity of the approximation, and which reveals the structure of a 'vortical layer'. It will be found, however, that the circumferential velocity and the pressure, and hence the lift and drag, given previously in equations (2.8) to (2.10) remain valid up to, and including, second order in ϵ and angle of attack.

Because of the fundamental nature of the problem which underlies both Stone's theory and the present work, the perturbation procedure associated with small yaw angle alone will first be studied in §§ 3.2 and 3.3, and consideration of the hypersonic shock layer will be made later as a simple extension of the analysis.

Inspection of the governing equations shows that the basic difficulty of the analysis lies in the last of the differential equations (2.3), namely that governing the entropy function. For small σ, the circumferential velocity σw is proportional to the yaw parameter σ. In Stone's analysis, as well as in the treatment given in § 2, the term associated with v is always considered to be of an order lower than the term associated with w in the same equation, and this leads to a solution for S and the related quantities which is not valid in the neighbourhood of the surface. Here it is essential to recognize that the small-perturbation procedure in Stone's theory overlooks the singular behaviour of S, which results from the vanishing of the normal velocity v at the surface. This point may be made more evident by writing the differential equation governing S in the form

$$\left[\theta\frac{\partial}{\partial\theta}-\sigma f(\theta,\omega;\,\sigma)\frac{\partial}{\partial\omega}\right]S = 0, \tag{3.1}$$

where the function f is defined as

$$f(\theta,\omega;\,\sigma) \equiv -\frac{w\theta}{J(1+\epsilon\theta)\,v}. \tag{3.2}$$

In view of the properties of v and w, the function f will generally be of unit order and non-vanishing, and the line $\theta = 0$ is clearly a line of singularities. In order to account for the behaviour of S in the neighbourhood of the body, the higher-order term $\sigma f\,\partial S/\partial\omega$ must be retained.

The general behaviour of S could be inferred from the characteristic equation associated with (3.1)

$$\frac{d\omega}{d\theta} = -\sigma\frac{f(\theta,\omega;\,\sigma)}{\theta}. \tag{3.3}$$

However, an explicit solution to (3.1) is not possible due to the lack of knowledge of an integrating factor for the characteristic equation (3.3). One must note that, as long as w and v are not exactly known, the general form of the function $f(\theta,\omega;\,\sigma)$ must be kept.

The second term in (3.1) associated with f is nevertheless important only in the vicinity of the surface $\theta = 0$, in so far as the first-order yaw effect is concerned. In the modified scheme to be used herein, the function $f(\theta,\omega;\,\sigma)$ in (3.1) will therefore be replaced by $f(0,\omega;\,0)$. For the determination of this expression, only the leading (zero) approximations of v and w at the surface are actually required. One may tentatively assume that the leading approximations for v and w are not affected by the singularity at the surface, which must be verified *a posteriori*.

According to this scheme, the first-order solution to the entropy field will be determined from the differential equation

$$\left[\theta\frac{\partial}{\partial\theta}-\sigma f(0,\omega;\,0)\frac{\partial}{\partial\omega}\right]S=0. \tag{3.4}$$

The boundary condition for S can be written to the same degree of accuracy into the form

$$S(\theta_0,\omega;\,\sigma)=S_0+\sigma g(\omega),\quad -\tfrac{1}{2}\pi<\omega<\tfrac{1}{2}\pi, \tag{3.5}$$

where the constants θ_0 and S_0 represent the shock position and entropy function at zero yaw, respectively. The function S can now be obtained readily from (3.4) by a separation of the variables. It is essential, however, to first examine the error in such a scheme of approximation, so that an unambiguous conclusion regarding the uniform validity of the approximation can later be drawn.

3.2. *The error in the modified approximation*

In the subsequent analysis of the approximation, the following properties of the functions f and g will be assumed:

(1) f and $\partial f/\partial\sigma$ exist and are non-vanishing except in the plane of symmetry (i.e. at $\omega=\pm\frac{1}{2}\pi$);

(2) as $\theta\to 0$, f approaches $f(0,\omega;\sigma)$ like θ^{ν}; as $\omega\to-\frac{1}{2}\pi$, f approaches zero like $(\omega+\frac{1}{2}\pi)^{\mu}$; and as $\omega\to\frac{1}{2}\pi$, f vanishes like $(\omega-\frac{1}{2}\pi)^{\lambda}$; where ν, μ and λ are positive constants;

(3) $dg/d\omega$ exists.

The properties enumerated above are consistent with the definitions given for f and g, and are sufficiently general to permit study of a cone at small yaw angle. It will be shown presently that, except in the neighbourhood of the stagnation streamline on the lee side $\theta=0$, $\omega=\frac{1}{2}\pi$, the approximation for S determined by (3.4) and (3.5) is uniformly valid to the first order provided the leading (zero) approximation of f, hence of v and w, are correctly specified at the surface. The latter provision will be checked subsequently in § 3.4.

For the present purpose, let $\bar{S}$ denote the approximate solution determined from (3.4) and (3.5). Thus

$$\bar{S}=S_0+\sigma g(\bar{\Omega}), \tag{3.6}$$

where $\bar{\Omega}$ is the characteristic variable pertaining to the approximate partial differential equation (3.4), which is related to θ and ω through

$$\theta^{\sigma}\exp\int_{\bar{\Omega}}^{\omega}\frac{d\eta}{f(0,\eta;\,0)}=\theta_0^{\sigma}\approx 1. \tag{3.7}$$

To study the error $R\equiv S-\bar{S}$, the differential equations governing S and $\bar{S}$, i.e., equations (3.1) and (3.4), will be combined to give

$$\left[\theta\frac{\partial}{\partial\theta}-\sigma f(\theta,\omega;\,\sigma)\frac{\partial}{\partial\omega}\right]R=\sigma[f(\theta,\omega;\,\sigma)-f(0,\omega;\,0)]\frac{\partial\bar{S}}{\partial\omega}. \tag{3.8}$$

The characteristics of the equation for R are the same as those of the exact equation (3.2). Since along the characteristics

$$\frac{d\theta}{\theta}=-\frac{d\omega}{\sigma f(\theta,\omega;\,\sigma)}=\frac{dR}{\sigma[f(\theta,\omega;\,\sigma)-f(0,\omega;\,0)]\,\partial\bar{S}/\partial\omega}, \tag{3.9}$$

the error R at an arbitrary point (θ_1, ω_1) can be evaluated, with the aid of the knowledge of $\bar{S}$ and (3.9), as

$$R(\theta_1, \omega_1) = \sigma^2 \int_{\theta_0(\Omega=\text{constant})}^{\theta_1} \frac{f(\theta, \omega;\, \sigma) - f(0, \omega;\, 0)}{\theta} g'(\bar{\Omega}) \frac{f(0, \bar{\Omega};\, 0)}{f(0, \omega;\, 0)} d\theta. \quad (3.10)$$

Here Ω is the characteristic variable for the exact solution, therefore not the same as $\bar{\Omega}$. To prove that R is of order σ^2, it is sufficient to show that the line integral along a curve of constant Ω on the right of (3.10) remains bounded. This is accomplished in two steps, which exclude in turn the neighbourhoods of the lines $\theta = 0$ and $\omega = \frac{1}{2}\pi$, as shown in figure 4.

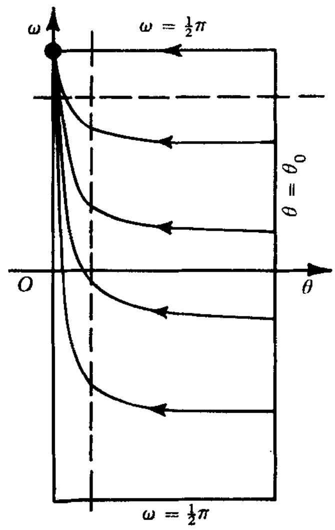

FIGURE 4. Illustration of the region of uniform validity for the approximate solution based on the modified scheme, showing exclusion of the neighbourhood of the stagnation streamline on the lee side, that is, $\theta = 0$, $\omega = \frac{1}{2}\pi$.

Since $g'(\omega)$ exists, and both $f(0, \omega;\, 0)$ and $f(\theta, \omega;\, \sigma)$ behave near the plane of symmetry $\omega = \frac{1}{2}\pi$ like $(\omega - \frac{1}{2}\pi)^\lambda$, the integrand of the integral in (3.10) is bounded everywhere except near the surface $\theta = 0$. Thus, excluding the neighbourhood of $\theta = 0$, R will remain uniformly of order σ^2.

To prove that the line integral is finite even when θ comes close to zero, the neighbourhood of $\omega = \frac{1}{2}\pi$ will now be excluded for (θ_1, ω_1). Then, the ratio $f(0, \bar{\Omega};\, 0)/f(0, \omega;\, 0)$ cannot become infinite. For, $f(0, \omega;\, 0)$ can now vanish only at $\omega = -\frac{1}{2}\pi$; but f is never negative and the characteristic equation (3.9) therefore requires that $\bar{\Omega} \leqslant \omega \leqslant \omega_0$. This is evident from figure 4. It follows that $f(0, \bar{\Omega};\, 0)$ will vanish earlier than $f(0, \omega;\, 0)$ when $\omega = -\frac{1}{2}\pi$ is approached. It remains to show that the integral

$$\int_{\theta_0(\Omega=\text{constant})}^{\theta_1} |f(\theta, \omega;\, \sigma) - f(0, \omega;\, 0)| \frac{d\theta}{\theta}$$

is finite for all points (θ_1, ω_1) sufficiently removed from the line $\omega = \frac{1}{2}\pi$. Since f will approach $f(0, \omega;\, \sigma)$ like θ^ν, the problem reduces to checking whether

$$\text{constant} \int_{\theta_0}^{\theta_1} \theta^{\nu-1} d\theta + \int_{\theta_0(\Omega=\text{constant})}^{\theta} \left| \frac{f(0, \omega;\, \sigma) - f(0, \omega;\, 0)}{\theta} \right| d\theta$$

remains bounded. Since $\nu > 0$, the first integral exists, and the second integral, in view of the characteristic equation, can be changed to

$$-\int_{\Omega(\Omega=\text{constant})}^{\omega_1} \left| \frac{f(0,\omega;\sigma)-f(0,\omega;0)}{\sigma f(\theta,\omega;\sigma)} \right| d\omega.$$

But the numerator of the integrand is of the order of σf_σ, or of σ. Meanwhile, since the f's approach zero either like $(\omega-\frac{1}{2}\pi)^\mu$, or like $(\omega+\frac{1}{2}\pi)^\lambda$, the integrand will not be infinite, and the integral therefore is of unit order. Hence, by excluding the neighbourhood of $\omega = \frac{1}{2}\pi$, the error is seen to be of order σ^2.

It follows from the two steps of the discussion given above that the approximate solution $\bar{S}$ is uniformly valid over the entire region $0 \leqslant \theta \leqslant \theta_0$, $-\frac{1}{2}\pi \leqslant \omega \leqslant \frac{1}{2}\pi$, except perhaps in the vicinity of the conical ray $\theta = 0$, $\omega = \frac{1}{2}\pi$. The error incurred by this approximate scheme is of order σ^2. The inability of the present scheme to deal with the singularity at $\theta = 0$, $\omega = \frac{1}{2}\pi$ is understandable from the illustration of figure 4. In view of the crowding of the characteristics (i.e. the projected streamlines) in the vicinity of this point, a small error in the slope of a characteristic could result in a serious discrepancy in the value for $\bar{S}$.

3.3. *Correction to Stone's theory*

According to the linearized theory of a circular cone at small yaw (Stone 1948), the circumferential velocity varies around the surface like $\cos\omega$, and hence

$$f(0,\omega;0) = A\cos\omega. \tag{3.11}$$

Meanwhile, the entropy function behind the slightly perturbed shock gives

$$g(\omega) = -B\sin\omega. \tag{3.12}$$

The constants A and B are both positive and generally of unit order, and may be related directly to the numerical quantities given in Kopal's table (1947 *b*).†
In terms of these two constants, the solution to the entropy field, which has taken the singularity at the surface into account, can now be expressed by

$$S = S_0 + \sigma B(1-\zeta^2)/(1+\zeta^2) + O(\sigma^2), \tag{3.13}$$

where‡

$$\zeta \equiv \theta^{A\sigma}\tan(\tfrac{1}{2}\omega+\tfrac{1}{4}\pi). \tag{3.14}$$

The result (3.13) possesses all the essential features of an entropy field around a yawed cone. In particular, it reveals clearly the structure of the 'vortical layer' discussed previously by Ferri (1950). On the body, $\theta = 0$, and the characteristic variable ζ vanishes; therefore the entropy is seen to be circumferentially uniform. At a small distance from the surface, since $A\sigma$ is small, and $\zeta \approx \tan(\frac{1}{2}\omega+\frac{1}{4}\pi)$, the entropy function becomes readily a function of ω alone, as given by the Stone theory. Furthermore, the projected streamlines, which are given by the contours of constant ζ, now appear to turn rather abruptly in the neighbourhood of the

† In terms of the variables $z, x, V, \eta/\bar{p}$ and $\xi/\bar{\rho}$ in Kopal's tables (1947 *b*),

$$A \equiv \left.\frac{z+2x/\sin\tau}{\partial V/\partial\vartheta}\right|_{\vartheta=\tau}, \quad B \equiv \left.\frac{\sin\tau[(\eta/\bar{p})-(\gamma\xi/\bar{\rho})]}{S_0}\right|_{\vartheta=\vartheta_{sh}}.$$

‡ Alternatively, $\zeta = \theta^{A\sigma}\{(1+\sin\omega)/(1-\sin\omega)\}^{\frac{1}{2}}$.

cone instead of terminating at the surface and to converge at the stagnation streamline on the lee side (i.e. at $\theta = 0$, $\omega = \frac{1}{2}\pi$). The layer in which the entropy abruptly changes is governed by the parameter $A\sigma$, and, in view of (3.14), the thickness appears to be considerably smaller than $O(A\sigma)$ which was estimated by Ferri (1950).

Since a high entropy gradient gives rise to a strong vorticity, the same behaviour is to be found in the velocity field of u. Meanwhile, in view of the relation $\rho^\gamma = p/S$, singularity of a similar nature also arises in the density field.

On the other hand, the linear yaw effects on the pressure, the circumferential and normal velocities, as well as the shock shape are not influenced by the vortical layer. This can be verified by substituting back into (2.3) the density field deduced from (3.13), and determining once again the solution for p, v and w. It is rather interesting to observe that the results obtained this way for p, v and w differ from Stone's merely in the addition of terms of the type

$$\sigma\theta[1-\theta^{A\sigma}].F(\omega), \quad \sigma\theta^2[1-\theta^{A\sigma}]\,G(\omega), \tag{3.15}$$

which are at most of order σ^2. Therefore, the use of the leading (zero) approximation for v and w in the calculation of $f(0, \omega; 0)$ is justified, and with the exclusion of the neighbourhood of the rearmost conical ray at the surface, the results based on the modified scheme indeed constitute a valid first-order approximation to the yawed-cone problem.

3.4. *The vortical layer in the shock-layer theory*

In the preceding discussion pertaining to supersonic flow, the very existence of the vortical layer is dependent on the small yaw angle; i.e. $\sigma \ll 1$. The subsequent results obtained under the hypersonic condition of (2.6) will show, on the other hand, that the vortical layer will appear in the form of a thin sublayer of the shock layer, and will be governed by the product $\epsilon\sigma$. Thus a rather thin vortical layer can exist, even if the yaw angle is not so small.

According to (2.8), the circumferential velocity approaches a very small value at the base of the shock layer. Namely, the non-vanishing, leading approximation of w at the surface $\theta = 0$ is

$$w(0,\omega) = \epsilon(1+\kappa)\cos\omega. \tag{3.16}$$

In order to account for the singularity at the surface due to yaw, one has now to take

$$f(0,\omega;\,0) = \epsilon(1+\kappa)\sec\tau\,\cos\omega, \tag{3.17}$$

in spite of the fact that ϵ itself is a small parameter. The entropy function may then be obtained explicitly as

$$\frac{p}{\rho^\gamma} = 1+\kappa+\epsilon\kappa+2\sigma\cos\tau\frac{1-\zeta^2}{1+\zeta^2}, \tag{3.18}$$

where $$\zeta \equiv \theta^{\sigma\epsilon(1+\kappa)\sec\tau}\tan\left(\tfrac{1}{2}\omega+\tfrac{1}{4}\pi\right),$$

a result which is essentially of the same form as (3.13). The error in (3.18) can be studied in much the same manner as in § 3.2, and, in the study, the vicinity of the stagnation streamline on the lee side must again be excluded. Equation (3.18) shows that the vortical layer appears in the form of a thin sublayer of the shock layer, and is now controlled not by σ alone but by the ratio of the yaw angle to the compression ratio across the shock. Hence, the vortical layer may still be quite thin even when the yaw angle is not very small. To illustrate this point, figure 5 gives the inviscid entropy field around a 45° (half-angle) cone at 17° of yaw, with a free-stream Mach number of 9·5, for a γ of 1·40. The contours of constant entropy, which also provide the pattern of projected streamlines, are calculated according to (3.18).

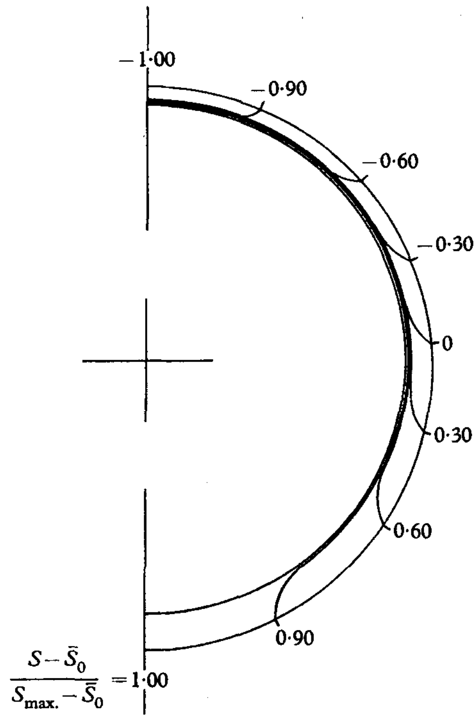

FIGURE 5. Illustration of the inviscid entropy field around a 45° (half-angle) cone at 17° of yaw, with a free-stream Mach number of 9·5, for a γ of 1·40. Contours of constant entropy also provide the pattern of streamlines projected in the ϑ–ω surface. $\gamma = 1{\cdot}40$, $\tau = 45°$, $\alpha = 17°$, $M_\infty = 9{\cdot}5$.

Assuming, tentatively, that the singular behaviour in the entropy field does not affect the leading approximations of v, w and the first-order correction to p, the density is readily obtained from (3.18) as

$$\rho \equiv \epsilon\frac{\rho_*}{\rho_\infty} = \frac{1}{1+\kappa}+\epsilon\left[\frac{1}{4}+\left(\frac{\kappa}{1+\kappa}\right)^2-\left(\frac{\theta}{1+\kappa}\right)^2+\frac{2}{1+\kappa}\ln(1+\kappa)\right.$$
$$\left.-\frac{2\sigma\cos\tau}{(1+\kappa)^2}\left[\frac{1-\zeta^2}{1+\zeta^2}+(1+\kappa)\sin\omega\right]\right], \quad (3.19)$$

and the radial velocity, from the Bernoulli relation, as

$$u \equiv \frac{u_*}{U_\infty} = \cos\tau-\tfrac{1}{2}\epsilon(1+\kappa)\sin\tau\,\tan\tau-\sigma\sin^2\tau\frac{1-\zeta^2}{1+\zeta^2}. \quad (3.20)$$

These results would agree with those given in § 2, if one let $\theta^{\epsilon(1+\kappa)\sigma \sec \tau} \to 1$. The vortical layer in the u-field is noteworthy in that it gives, to the first order in ϵ and σ, a uniform velocity at the body surface.

Using the corrected results for ρ and u and the differential equation (2.3), the effect of the vortical layer can be shown to belong only to the third order of $(\epsilon+\sigma)$ in the pressure field, and to the second order of $(\epsilon+\sigma)$ in v and w. In other words, except for the density ρ and the radial velocity u, and with the exclusion of the vicinity of the stagnation streamline on the lee side, the foregoing analysis has confirmed the validity of the solutions obtained previously in § 2 for the pressure p, and the velocities v and w. With the help of these approximate solutions, the values of w, u and ρ at the surface can in fact be determined consistently to an even higher order by direct integration of the differential equation (2.3) along the cone surface.

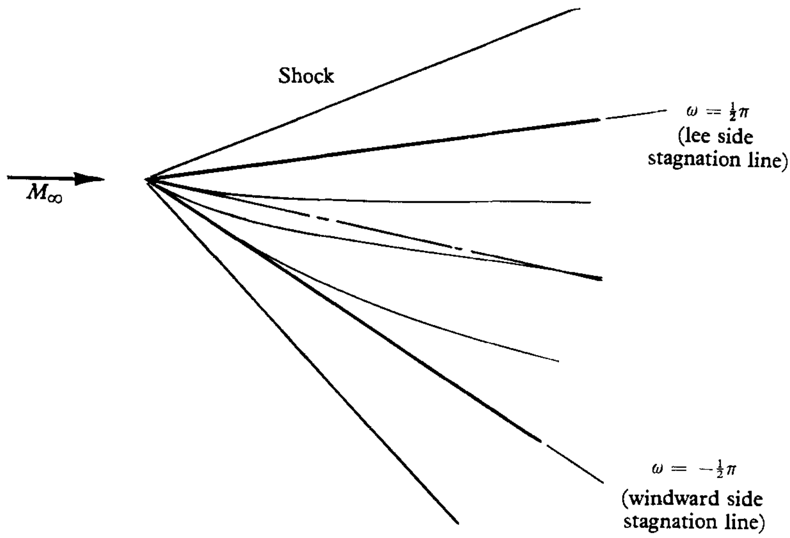

FIGURE 6. Pattern of inviscid streamlines on the surface of a yawed cone in hypersonic flow.

$$\left(\frac{r}{r_0}\right)^{2(1+\kappa)\epsilon\alpha} \approx \tan\left(\tfrac{1}{2}\omega+\tfrac{1}{4}\pi\right).$$

To illustrate more clearly the flow pattern near the cone surface, figure 6 provides a qualitative description of the surface streamlines. As shown, all streamlines on the cone must come from the stagnation line on the windward side, so that entropy is constant on the cone surface. The angles between the streamlines and the surface conical rays are, however, of the order $\epsilon\alpha$. For small $\epsilon\alpha$, surface streams will follow closely the surface conical rays, and can be traced back to the immediate vicinity of the pointed nose. This pattern of surface streamlines may be described by the equation

$$\left(\frac{r}{r_0}\right)^{2(1+\kappa)\epsilon\alpha} \approx \tan\left(\tfrac{1}{2}\omega+\tfrac{1}{4}\pi\right), \tag{3.21}$$

where r_0 is the radial distance at which the surface streamline crosses the ray $\omega = 0$. The above equation implies, in fact, that all surface streamlines must come from the cone apex even when $\epsilon\alpha$ does not tend to zero.

From the results of the above study, certain important features concerning the flow behaviour near the base of the shock layers may be summarized. The circumferential velocity will, according to the analysis, approach a small value at the surface, being of the order $\alpha\epsilon(1+\kappa)\, U_\infty$ on the cone. Thus, when ϵ is sufficiently small, the inviscid streamlines near the surface will tend to follow conical rays from the apex in a manner quite independent of the yaw angle. As a result, a thin vortical layer appears at the base of the shock layer, which is characterized by large gradients of entropy, density and velocity, and its thickness is controlled by the product $\sigma\epsilon(1+\kappa)$. At the base of this vortical layer, not only is the entropy constant, as required by general considerations, but the flow speed is also found to remain circumferentially uniform, at least to the first-order approximation in $(\epsilon+\sigma)$. Quite evidently, the above conclusions concerning the flow field at the base of the shock layer are limited neither to small yaw angles nor to bodies of particular shapes. In view of its importance to the analysis of hypersonic boundary layer, the behaviour of inviscid, hypersonic flow near the surface of an arbitrary pointed body will be subsequently studied.

4. On shock layers in three dimensions

4.1. *Stiffness of streamlines*

Before studying the flow field near the body surface, two basic properties of the inviscid flow within a shock layer should be noted; namely, the persistence of flow speed (and enthalpy) along a streamline and the apparent stiffness of the streamline itself.

These two properties follow readily from the fact that fluid particles within a shock layer, by virtue of the high compression ratio across the shock, possess great momentum, and their motions are therefore rather unaffected by the tangential pressure gradients. As a result, the flow speed, and hence the enthalpy (which follows from the Bernoulli relation), as well as other related quantities, will remain essentially constant along the streamline. It also follows that the movement of a fluid element within the thin shock layer is identifiable as the motion of a particle which is constrained to move along the surface but is otherwise free. According to classical mechanics (Goldstein 1950), the trajectory of this motion must describe the shortest distance between successive points along the curve, that is, along a 'geodesic' of the surface. Hence, the streamlines within the shock layer appear to possess a certain degree of 'stiffness', so to speak, since they show little response to the transverse pressure gradient and tend to follow the same path as in force-free motion on a surface.

The two properties of the flow field described pertain to the leading approximation for a high compression ratio across the shock. The formal procedure for determining the complete streamline pattern and the surface pressure for an arbitrary body, which is not the subject of the present study, has been discussed in detail by Hayes & Probstein (1959), Guiraud (1959*a*) and Maikapar (1959).

4.2. *Pattern of surface streamlines on pointed bodies*

The inviscid streamlines in the immediate vicinity of the surface of a pointed body have another simplifying property. They are the geodesics originating from the apex of the pointed nose. This fact is suggested readily by the result of the analysis given in §§ 2 and 3 for a slightly yawed, circular cone. There, the circumferential velocity w_* at the surface is found to belong to the order $\alpha\epsilon(1+\kappa)U_\infty$. Hence, in the limit $\epsilon \to 0$, provided $\kappa = O(1)$, streamlines near the surface approach surface conical rays which are the geodesics from the apex. This pattern of streamlines is therefore completely determined by the surface geometry and is essentially independent of the inclination of the body and other free-stream conditions. In the following discussion, consideration will be given to non-slender pointed bodies of arbitrary shape.

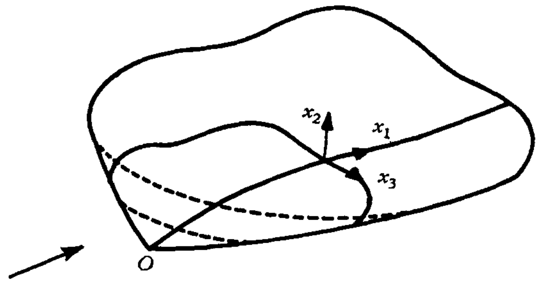

FIGURE 7. A system of orthogonal curvilinear co-ordinates in which x_1 is the distance measured along the geodesic originating from the apex, and x_2 along the surface normal.

In order to show that geodesic streamlines at the base of the shock layer originate from the vicinity of the pointed nose, the circumferential velocity on a conical surface will first be examined. It is convenient to use a system of orthogonal curvilinear co-ordinates as illustrated in figure 7, in which x_1 is the distance along the geodesic from the apex and x_2 the distance from the surface. The differential equation governing the circumferential velocity u_3, when applied directly on a conical surface, may be written as

$$u_3\left[\frac{\partial u_3}{\partial x_3} + u_1\,|G(x_3)|\right] = O[\epsilon u_1^2], \tag{4.1}$$

where $|G|$ is the function which appears in the expression for an arbitrary line element $d\mathscr{S}$ on the conical surface, $d\mathscr{S}^2 = dx_1^2 + x_1^2 G^2(x_3)\,dx_3^2$, and its magnitude can in most cases be regarded roughly as the body thickness. Underlying (4.1) is, of course, the assumption of conical symmetry which is valid under the shock-layer approximation provided the shock or the body angle is not too close to 90°. Also implicit is a requirement similar to that on κ in §§ 2 and 3; that is, the compression ratio across the shock must be sufficiently strong that the estimate given on the right-hand side of (4.1) holds. For a non-slender body, G belongs to unit order and (4.1) shows that u_3/u_1 is of order ϵ. Therefore, inviscid surface streamlines will approach conical rays generating the surface, as $\epsilon \to 0$.

The extension to an arbitrary pointed body follows immediately: near the nose, the flow field approaches a conical structure, and, according to the preceding discussion, the limiting streamlines must follow conical rays from the apex. Continuation of these geodesics from the nose region to the remaining portion of the body provides the complete streamline pattern on the pointed body.

For the streamline pattern on slender bodies, however, two situations must be considered. In one case, while the body is slender, the cross-flow is not so strong that the surface streamline will still retain the same pattern as on a non-slender pointed body. In the other case, the cross-flow field is strong enough to offset the geodesic streamline pattern on the surface and the flow will approach that of an infinite cylinder at finite yaw angle. For a slender circular cone at yaw, a criterion for this sort of classification can be obtained from the magnitude of the parameter $\epsilon\sigma(1+\sigma)$. When this parameter is of order unity or less, one has $u_3/U_\infty = O[\epsilon\sigma(1+\sigma)]$, which agrees with the previous result; and when it is larger than unit order, one has $u_3/U_\infty = O[\{\epsilon\alpha(\tau+\alpha)\}^{\frac{1}{2}}]$, which, when interpreted with the aid of the cross-flow concept, is consistent with the results of Chester (1956) and Freeman (1956) for the bluff bodies. In either situation, nevertheless, a vanishing u_3 results in the limit of $\epsilon \rightarrow 0$.†

4.3. *Thin vortical layer on three-dimensional pointed bodies*

For a circular cone at small yaw angle, the analysis in the previous section reveals that a thin vortical layer exists at the base of the shock layer, across which an abrupt adjustment in enthalpy and flow speed occur. This 'vortical' layer is controlled not only by the yaw parameter σ, which characterizes the cross-flow effect in general, but also by the product $\epsilon\sigma$. The vortical layer is indeed a characteristic feature of shock layers on three-dimensional pointed bodies, as will be seen to follow readily from the streamline pattern peculiar to the surface of a pointed body.

In order to perceive the connexion between the surface streamline pattern and the sublayer-like behaviour of the flow field in the general case, one may first study the projected streamlines in a flow field having conical symmetry. The trajectory of a fluid particle, projected on a spherical screen of unit distance from the apex (i.e. the ϑ–ω plane), is determined by the ratio of the normal and the circumferential velocities, u_2/u_3. The normal velocity component u_2 must vanish at the surface. While the circumferential velocity u_3 does not vanish at the surface, except in the plane of symmetry (and perhaps in certain other isolated regions), it nevertheless approaches a rather small value at the surface, being in most cases of order ϵ. Thus, at a point removed from the body surface, the projected streamline has a non-vanishing slope (with respect to the body surface). In approaching the cone surface, the slope diminishes with u_2 and the trajectory becomes eventually tangent to the surface. However, by virtue of the fact that the circumferential velocity on the surface takes on a small value, effective adjustment of the slopes of these projected streamlines does not take place except at the immediate vicinity of the cone surface. The abrupt changing of the particle paths near the surface then gives rise to a sublayer-like behaviour. In view of the fact that the body surface is wetted, so to speak, only by the streamlines coming from the strongest part of the shock, and on account of persistency, the flow speed (and enthalpy) at the base of the vortical layer must be circumferentially uniform.

† However, representation of the surface-streamline pattern by the geodesics from the pointed nose is not always satisfactory for the slender bodies.

Once the existence of a thin vortical layer has been established in the conical region (i.e. the vicinity of the pointed nose) the question of its existence over the remaining part of the body is answered, since enthalpy and flow speed will persist along neighbouring streamlines downstream of the conical region. In fact, along any geodesic from the apex, one will find the 'jump' in enthalpy or flow speed across the vortical layer unchanged.

The flow structure of the vortical layer in the general case is very similar to that illustrated previously for the circular cone at small yaw angle. In fact, a treatment of the entropy field quite analogous to the scheme used in § 2 may be adopted for the more general situation. In the case of a circular cone at large yaw angle, for example, one may study, instead of (3.4),

$$\left[\theta\frac{\partial}{\partial\theta}+\epsilon F(\omega)\frac{\partial}{\partial\omega}\right]S = 0, \tag{4.2}$$

where $F(\omega)$ is the (non-vanishing) leading approximation of $(\sigma w\theta)/(\epsilon v\cos\tau)$ at the surface, which does not require a knowledge of the vortical layer effect.

5. Concluding remarks

A detailed treatment of the problem of inviscid hypersonic flow over yawed circular cones is given using the shock-layer approximation in conjunction with small perturbations in the yaw angle. The singularity of the entropy field is examined, and a uniformly valid solution is obtained which is applicable to hypersonic as well as to supersonic régimes. In the hypersonic case, the solution provides an explicit simple description of the shock layer and exhibits a thin vortical layer at the inner edge of the shock layer. Within the framework of the shock-layer theory, certain features of the flow field around a pointed, but otherwise arbitrary, body are studied. At the inner edge of the shock layer, the streamlines are found to follow the surface geodesics originating from the pointed nose, and a thin vortical layer is shown to exist.

Of particular interest among the results are the formulas describing the entropy field of a circular cone at small angle. In order to account for the singularity at the surface, which was overlooked in the previous analysis by Stone (1948, 1952), a modified scheme of small perturbation (with respect to yaw angle) is used. The solution obtained reveals the structure of a vortical layer as anticipated by Ferri (1950), although its thickness appears to be considerably less than Ferri's estimate. Except in the neighbourhood of the stagnation streamline on the lee side, this scheme is shown to give a valid first-order approximation to the entropy field. In general, the method provides corrections to Stone's theory for the density and the radial velocity fields; it also confirms the validity of the first-order results for the pressure field, and the field of circumferential velocity computed by Kopal (1947*b*).

In the hypersonic régime where shock-layer theory applies, the modified scheme provides approximations to the flow fields which are generally, and uniformly, valid to the first order in $\epsilon = (\gamma-1)/(\gamma+1)$ and in the angle of attack. For the pressure and the circumferential velocity fields, valid formulas have, in fact, been obtained consistently to the second order, including the non-linear yaw effect. Noteworthy among these results is the small value of the circumferential velocity near the surface, which is seen to belong to the order $\epsilon\alpha U_\infty$. Since ϵ is assumed to be small, this means that the streamline at the inner edge of the shock layer must follow closely the generator of the cone surface in a manner quite independent of the yaw angle. On this account a vortical layer appears in the form of a sublayer at the inner edge of the shock layer. Hence, this vortical layer is rather thin (being less than $\alpha\epsilon^2$ in angular thickness) even when the yaw angle is not very small.

The explicit results of the foregoing analysis have thus illustrated clearly certain particular features of the shock layer which should be meaningful to the analysis of boundary layers in three dimensions. In the light of these results, a study is made for the more general situation in which the specific assumption of small yaw angle as well as that of a circular cone are not required. In exploring the three-dimensional shock layer, two basic flow properties are noted. Namely, that flow speed and enthalpy persist along the streamline, and that fluid particles tend to travel along the shortest path on the body surface; i.e., along the surface geodesics. The last fact implies that the geometry of the streamline is rather insensitive to the effect of the transverse pressure gradient and the streamlines thus appear to be rather 'stiff'. The fact that the streamlines are geodesics does not define fully the streamline pattern. At the surface of a pointed body, however, the (inviscid) streamline geodesics do approach a definite pattern. They belong to the family of geodesics originating from the pointed nose, irrespective of the angle of attack. As a result of the streamline pattern on the surface, a thin vortical layer will generally appear at the inner edge of the shock layer around a three-dimensional pointed body. At the base of the vortical layer, not only is the entropy constant, but also the flow speed (and enthalpy) must be essentially uniform.

The conclusions concerning the streamline pattern on a pointed body and the sublayer behaviour of the flow field that follows are borne out by the detailed results of the foregoing analysis for a circular cone, although the latter study has been restricted to small yaw angle. One may note that, for bodies which are extremely slender, the family of geodesics emanating from the pointed nose cannot always be used to provide an adequate description of the inviscid streamline pattern on the body surface. However, the transverse velocity on the surface may still be regarded as being small and a vortical layer can still be found, although it would be less pronounced.

The implication of the foregoing study for the analysis of boundary layers in hypersonic flow should not be overlooked. Since the inviscid streamlines near the surface of a pointed body closely follow the geodesics originating from the pointed nose, the streamlines in the boundary layer must follow the same pattern, if the 'stiffness' property can also be maintained there. The latter condition may indeed be fulfilled in certain classes of hypersonic boundary layers, particularly those on non-slender pointed bodies in which the density level is generally the same as, or even higher than, that of the outer flow (Cheng 1961). Hence, the presence of a thin shock layer around a pointed body may generally imply a small secondary flow in the boundary layer. The boundary-layer streamlines in this instance will be determined by the body geometry alone and are independent of the angle of attack of the body. The geodesic co-ordinates (as illustrated in figure 7) thus appear to be the natural choice for analysing hypersonic boundary layers in three dimensions.

Meanwhile, the existence of a thin vortical layer indicates the importance of outer-flow vorticity for the boundary-layer development in hypersonic flow. One recalls that the vortical layer in the supersonic case is associated primarily with a small cross-flow, hence its presence in any case is not too important. The vortical layer that appears in the hypersonic shock layer, on the other hand, should have a far greater effect because of the much larger variation in the outer entropy field. However, on account of the rather small thickness of the vortical layer, as revealed by the foregoing analysis on a circular cone, its effects may perhaps be disregarded under certain practical circumstances, namely when the boundary layer itself has an appreciable thickness. None the less, should the Reynolds number indeed be very high, the outer edge of the boundary layer must then be found at the base of the vortical layer. In this limiting case, the flow speed at the boundary-layer outer edge may be taken simply as a constant. A rather involved, but otherwise more interesting problem lies in the intermediate case, in which the boundary layer and the highly vortical inner region of the shock layer must be matched properly. The inviscid solution for the yawed circular cone obtained herein, which provides an analytical description of the structure of the vortical layer, may perhaps serve as a starting-point for such an inquiry.

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